CHARACTERIZATION OF RECTIFIABLE MEASURES THAT ARE CARRIED BY LIPSCHITZ GRAPHS

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ABSTRACT. The Analysts' Traveling Salesman Problem asks for necessary and sufficient conditions under which a set is contained inside of a Lipschtiz image. One direction for further study is to find a characterization of measures carried by Lipschitz graphs. In previous work, balls centered at each point in the support are used to give a characterization of doubling measures that are carried by Lipschitz graphs. To further extend that work, we develop and prove sufficient and necessary conditions for doubling measures carried by Lipschitz graphs in terms of dyadic cubes. Along the way, we prove a doubling measure property and a geometric lemma for measures that hold under the dyadic cube regime. These new results provide a characterization of measures carried by Lipschitz graphs that is more discrete in nature.

1. INTRODUCTION

A general goal of geometric measure theory that is to understand the global characterization of a measure through the geometric data. In this paper, we contribute to the goal by studying the interaction of measures with graphs. To formalize the notion and dive into this paper, we introduce the terminology below.

Definition 1.1 (Lipschitz Function). *The function f is called Lipschitz if*

$$|f(x)-f(y)| \leq c|x-y| \quad (x, y \in X)$$

for $0<c<\infty$

Definition 1.2 (Lipschitz Graph). *Let $V \in G(n, m)$ be an m-dimensional plane in $\mathbb{R}^n$, and $V^{\perp} \in G(n, n-m)$ denotes its orthogonal complement. Let $f: V \rightarrow V^{\perp}$ be a Lipschitz function. Then* $\operatorname{Graph}(f)=\{(x, f(x)): x \in V\}$ *is an m-Lipschitz graph.*

where Grassmannian G denotes a collection of linear planes in $\mathbb{R}^n$. For example, $V \in G(1,2)$ represents the line in $\mathbb{R}^2$ going through the origin which is what we mainly work with in this paper.

Definition 1.3 (Carried). *Let $(\mathbb{X}, \mathcal{M})$ be a measurable space, and let $\mathcal{N} \subset \mathcal{M}$ be a family of measurable sets. We say μ is carried by $\mathcal{N}$ if there exist countably many $N_i \in \mathcal{N}$ such that* $\mu\left(\mathbb{X} \backslash \bigcup_i N_i\right)=0$

In our scenario, when we say μ is carried by m-Lipschitz graphs, it is equivalent that there exists a subset of $\mathbb{R}^n$ is contained μ-a.e. in an m-Lipschitz graph.

Recently, [BS15] and [BS17] characterized rectifiable Radon measures on $\mathbb{R}$ by L^2 Jone's beta numbers, and [Nap20] characterized the measures in terms of rectifiable graphs. However, the basic schema and proof they used for characterization is under the continuous space, restricted in balls. In this paper, we introduce the idea of characterization of rectifiable measures under the

Date: September 5, 2021.
Key words and phrases. rectifiable measures, dyadic cubes, bad cones, K-doubling measures, Lipschitz Graph.

discrete space of dyadic cubes with fixed generations. Some related work for rectifiable measure involved with the dyadic cubes can be found in [Jon90], [Tol09], [BS16], and [BN21].

Definition 1.4 (Dyadic Cubes). *A dyadic cube Q is a set of the form*

$$Q := \left[\frac{j_1}{2^k}, \frac{j_1+1}{2^k}\right) \times \cdots \times \left[\frac{j_n}{2^k}, \frac{j_n+1}{2^k}\right), \quad k, j_1, \ldots, j_n \in \mathbb{Z} \tag{1}$$

For convenience, we let $\operatorname{side} Q$ denote the side length of the dyadic cube Q, and $\operatorname{center} Q$ denote the center point of Q. The dilation cube of Q then is defined by $nQ, n \in \mathbb{N}$ where $\operatorname{side} nQ = n \operatorname{side} Q, \operatorname{center} nQ = \operatorname{center} Q$. In some of proofs in this paper, we used Q_k to denote the dyadic cube with side length $|2^{-k}|$.

Definition 1.5 (Doubling Measure). *We say that a measure μ is doubling measure if there exists a constant K such that for all $r > 0$ and μ-a.e. x,*

$$\mu(B(x, 2r)) \leq K\mu(B(x, r))$$

In this case, we say μ is $K-$doubling.

2. Preliminaries

In this paper, we extend the results of [Nap20] under the dyadic cube system. We will first bring the point bad cones which are had for characterize Lipschitz graphs.

Definition 2.1 (Bad Cones). *Let V be an m-dimensional plane in $\mathbb{R}^n$, $0 < \alpha < \infty$, then we can define the bad cone at x with respect to V and α by:*

$$C_{\mathcal{B}}(x, V, \alpha) := \{y \in \mathbb{R}^n : \operatorname{dist}(y - x, V) > \alpha|x - y|\}$$

For convenience, we let $C_{\mathcal{B}}(x, r, V, \alpha) := C_{\mathcal{B}}(x, V, \alpha) \cap B(x, r)$.

Based on the definition of bad cones, we can define good cones at x with respect to V and α as $C_{\mathcal{G}}(x, V, \alpha) := \mathbb{R}^n \setminus C_{\mathcal{B}}(x, V, \alpha)$. To be clear, We define the distance $\operatorname{dist}(X, Y)$ from a set X to another set Y by:

$$\operatorname{dist}(X, Y) := \inf \{|x - y| : x \in X, y \in Y\}$$

We now introduce the cones to the dyadic cube generations. Different from cones centred at balls, we define two kinds of cube cones based on intersection and union of point cones with centres inside the cubes. To be more specific about the definition of cones on the dyadic cube, we further split the definition of bad cone based on intersection and containment of their boundary with the dyadic cubes system.

Definition 2.2 (Bad Cube Cones). *Let Q be the dyadic cube, two basic definitions of a bad cone at Q with respect to an m-dimensional linear plane V and α are:*

$$C^1_{\mathcal{B}}(Q, V, \alpha) := \bigcup_{x \in Q} C_{\mathcal{B}}(x, V, \alpha) \quad \textit{and} \quad C^2_{\mathcal{B}}(Q, V, \alpha) := \bigcap_{x \in Q} C_{\mathcal{B}}(x, V, \alpha)$$

two discrete definitions of bad cube cones:

$$C^{k,1}_{\mathcal{B}}(Q, V, \alpha) := \{R : R \cap C^k_{\mathcal{B}}(Q, V, \alpha) \neq \emptyset\}, \quad C^{k,2}_{\mathcal{B}}(Q, V, \alpha) := \{R : R \subset C^k_{\mathcal{B}}(Q, V, \alpha)\}$$

for R denote the dyadic cube with the same side length as Q in the cube system, and $k = 1, 2$.

We extend the idea of [Nap20, Theorem D] for our main theorem, and the corollary below is useful to construct the proof.

Corollary 2.1 ([Nap20, Corollary 7.1]). *Let μ be a Radon measure on H, V be an m-dimensional linear plane in H, $\alpha \in (0,1)$, and fixed $0 < r < \infty$. If for μ-a.e. $x \in H$*

$$\mu\left(C_{\mathcal{B}}(x, r, V, \alpha)\right) = 0$$

then μ is carried by m-Lipschitz graphs.

In consideration of the dyadic cube generations, we will first extend the doubling measure property in the dyadic cube system.

Lemma 2.1. *Suppose that a doubling measure μ on $\mathbb{R}^2$ is $K-$doubling, then for any dyadic cube Q containing μ-typical point x, $s, n \in \mathbb{N}, n \geq 2$,*

$$\mu(2^n sQ) \leq K^{3n-3}\mu(2sQ) \tag{2}$$

Proof. By the half open definition of the dyadic cube, for μ-a.e. $x \in \mathbb{R}^2$, x must be in only one cube Q. Let $r_s = s \cdot \operatorname{side} Q/2$. We claim that $2^n sQ \subset B(x, 2^{3n-3}r_s)$ and $B(x, r_s) \subset 2sQ$. For any $q \in 2^n sQ$, $|\operatorname{center} Q - x| \leq \sqrt{2}/2 \cdot \operatorname{side} Q$ and $|\operatorname{center} 2^n sQ - q| \leq \sqrt{2}/2 \cdot \operatorname{side} 2^n sQ = 2^{n-1}\sqrt{2}s \cdot \operatorname{side} Q$. Note that $n \geq 2$, so

$$|x - q| \leq \left(\frac{\sqrt{2}}{2} + 2^{n-1}\sqrt{2}s\right)\operatorname{side} Q < \frac{2^{3n-3}s}{2}\operatorname{side} Q = 2^{3n-3}r_s$$

Then $q \in B(x, 2^{3n-3}r_s)$ and thus $2^n sQ \subset B(x, 2^{3n-3}r_s)$. Consider that for all $p \in \bigcup_{q\in Q} B(q, r_s) \setminus sQ$, $\operatorname{dist}(p, \partial sQ) \leq r_s = \operatorname{dist}(\partial 2sQ, sQ)$, so $B(x, r_s) \subset \bigcup_{q \in Q} B(q, r_s) \subset 2sQ$. Therefore, by the containment and doubling measure property,

$$\mu(2^n sQ) \leq \mu(B(x, 2^{3n-3}r_s)) \leq \prod_{i=1}^{3n-3} K \cdot \mu(B(x, r_s)) \leq K^{3n-3}\mu(B(x, r_s)) \leq K^{3n-3}\mu(2sQ)$$

□

Now we propose our main theorem here. We use the notation $Q \downarrow x$ to indicate we are interested in the dyadic cube Q with side length 2^{-k} containing x as $k \to \infty$.

Theorem A. *Let μ be a K-doubling measure on $\mathbb{R}^2$. Then for μ-a.e. $x \in \mathbb{R}^2$ contained in the dyadic cube Q, there exists $V \in G(1,2)$, $\alpha \in (0,1)$, and $s = s(\alpha) \in \mathbb{N}$ such that*

(i) (Sufficient Condition) if the limit

$$\lim_{Q \downarrow x} \frac{\mu(C_{\mathcal{B}}^{2,1}(Q, V, \alpha) \cap 2sQ)}{\mu(2sQ)} = 0 \tag{3}$$

then μ is carried by Lipschitz graphs with respect to V and the Lipschitz constant for each graph is at most $1 + 1/\sqrt{(1-\alpha'^2)}$ for any $\alpha < \alpha' < 1$.

(ii) (Necessary Condition) if μ is carried by Lipschitz graphs with respect to V and the Lipschitz constant at most $\sqrt{\alpha^2/(1-\alpha^2)}$, then (3) holds.

3. Proof of the Main Result

To begin with, we first provide some corollaries that will be used to fully prove the sufficient and necessary condition of Theorem A.

We will first propose a lemma for a distance that guarantees the distance for the containment among cubes.

Lemma 3.1. *For $0 < \alpha < \alpha' < 1$, $V \in G(1,2)$, a dyadic cube Q in $\mathbb{R}^2$, and $s \geq \frac{3\sqrt{2}(1+\alpha)}{\alpha'-\alpha}$. Let $2R$ be a dilation of a cube R such that R is the dyadic cube with the same side length as Q and*

$$\operatorname{dist}(Q, 2R) \geq s \cdot \operatorname{side} Q$$

then

(i) if $2R \subset C^{2,1}_{\mathcal{B}}(Q, V, \alpha')$, then $2R \subset C^{2,2}_{\mathcal{B}}(Q, V, \alpha)$.
(ii) if for $n = n(s) \in \mathbb{N}$ such that $2^n \geq 2(\sqrt{2}+1)s + 3\sqrt{2}$, then $2sQ \subset 2^n R$.
(iii) if for $m = m(s) \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}+2)s + 3\sqrt{2}$, then $3sQ \subset 2^m R$.

The proof can be found in the appendix.
With everything established, we are ready to prove Theorem A now.

Proof of Theorem A. We first show the sufficient condition holds. In order to distinguish the dyadic cube with different generations, we denote Q_k as dyadic cube with side length 2^{-k}, $k \in \mathbb{N}$. By (3) there exists a large enough k such that for any $\delta > 0$,

$$\mu(C^{2,1}_{\mathcal{B}}(Q_k, V, \alpha) \cap 2sQ) < \delta\mu(2sQ_k) \tag{4}$$

Fix a k and $Q_k \ni x$. Now choose $s \geq \dfrac{6\sqrt{2}(1+\alpha')}{\alpha - \alpha'}$, $s \in \mathbb{N}$. Then construct sets:

$$S_{Q_k} := \left\{2R_k : \operatorname{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \operatorname{side} Q_k, 2R_k \cap (2sQ_k \cap C^{2,1}_{\mathcal{B}}(Q_k, V, \alpha')) \neq \emptyset\right\}$$
$$S'_{Q_k} := \left\{2R_k : \operatorname{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \operatorname{side} Q_k, 2R_k \subset 2sQ_k \cap C^{2,1}_{\mathcal{B}}(Q_k, V, \alpha')\right\}$$
$$S''_{Q_k} := S_{Q_k} \setminus S'_{Q_k}$$

where $2R_k$ is the dilation cube of the dyadic cube R_k with the same side length as Q_k. Then for any $2R_k \in S'_{Q_k}$, by Lemma 3.1 (i), $2R_k \subset C^{2,2}_{\mathcal{B}}(Q_k, V, \alpha)$ and consequently $2R_k \subset C^{2,1}_{\mathcal{B}}(Q_k, V, \alpha) \cap 2sQ_k$ as well. Then, fix $2^n = n(s), n \in \mathbb{N}$ such that $2^n \geq (\sqrt{2}+1)s + 3\sqrt{2}$, then by Lemma 3.1 (ii), $2sQ_k \subset 2^n R_k$. Now assume that there are some μ-a.e. x are contained in $2R_k$, then by (4) and Lemma 2.1 (2),

$$\mu(2^n R_k) \leq K^{3n-3}\mu(2R_k) \leq K^{3n-3}\mu(C^{2,1}_{\mathcal{B}}(Q, V, \alpha) \cap 2sQ_k) < \delta K^{3n-3}\mu(2sQ_k)$$

which is a contradiction if we choose $\delta < K^{-3n+3}$. Thus,

$$\mu(S'_{Q_k}) = \mu(\bigcup_{2R_k \in S'_{Q_k}} 2R_k) = \sum_{2R_K \in S'_{Q_k}} \mu(2R_k) = 0$$

Similarly, we will consider any $2R'_k \in S''_{Q_k}$. Now fix $2^m = m(s), m \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2}/2 + 1)s + 3\sqrt{2}$ and then by Lemma 3.1 (ii), $2R'_k \subset 3sQ_k \subset 2^m R_k$. Then with the similar contradiction proof noting that $\mu(2sQ_k) \leq \mu(3sQ_k)$, we can say $\mu(S''_{Q_k}) = 0$. Hence, $\mu(S_{Q_k}) = \mu(S'_{Q_k} \cup S''_{Q_k}) = 0$. Note that this will hold for all dyadic cubes with the smaller side length than Q_k.

Assume that $r = (s - \sqrt{2}/2) \cdot \operatorname{side} Q_k$ then for μ-a.e. $x \in Q_k$, any $y \in C_{\mathcal{B}}(x, r, V, \alpha')$, we have $\operatorname{dist}(x-y, V) > \alpha'|x-y|$. Choose $k' > k$, and then there exists dyadic cube $Q_{k'} \ni x$ such that $(\sqrt{2} + s/2) \cdot 2^{-k'} \leq |y - x|$. Then $\operatorname{dist}(y, Q_{k'})| \geq |y-x| - \operatorname{dist}(x, \partial Q_{k'}) \geq (\sqrt{2} + s/2 - \sqrt{2}) \operatorname{side} Q_{k'} = s/2 \cdot \operatorname{side} Q_{k'}$, so that $y \in \bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k\}$. Hence,

$C_{\mathcal{B}}(x,r,V,\alpha') \subset \bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k\}$. Thus, for μ-a.e. $x \in E$ where x must be contained in only one Q_k,

$$\mu(C_{\mathcal{B}}(x,r,V,\alpha')) \leq \mu\left(\bigcup_{i=k}^{\infty}\{S_{Q_i} : x \in Q_i \subset Q_k)\}\right) = 0$$

It follows immediately that for μ-a.e. $x \in E$, $\mu(C_{\mathcal{B}}(x,r,V,\alpha')) = 0$. Applying Corollary 2.1, the sufficient condition holds then.

To prove the necessary condition, suppose that E is contained in a collection of Lipschitz graphs $\{\Gamma_i\}$ where $\Gamma_i \cap E = \{(v, f_i(v)) : (v, f_i(v)) \in E\} \subset V \times V^{\perp} = \mathbb{R}^2$ where f_i is a Lipschitz function $f_i : V \to V^{\perp}$ with Lipschitz constant L_i at most $\sqrt{\alpha^2/(1-\alpha^2)}$. Let any point $p = (v_1, f_i(v_1)), q = (v_2, f_i(v_2)) \in E \cap \Gamma_i$, then we have $|f_i(v_1) - f_i(v_2)| \leq L_i|v_1 - v_2|$. Consider the good cone $C_{\mathcal{G}}(p,V,\alpha)$, then,

$$|p-q| = \sqrt{|v_1 - v_2|^2 + |f_i(v_1) - f_i(v_2)|^2} \geq \sqrt{\frac{1}{L_i^2} + 1}|f_i(v_1) - f_i(v_2)| \geq \frac{1}{\alpha}\operatorname{dist}(p-q, V)$$

Thus, $q \in C_{\mathcal{G}}(p,V,\alpha)$. As p, q, Γ_i are arbitrary, $E \subset C_{\mathcal{G}}(x,V,\alpha), \forall x \in E$. Now, for $Q_k \ni x$,

$$C_{\mathcal{B}}^{2,1}(Q_k,V,\alpha) \cap 2sQ_k \subset C_{\mathcal{B}}(x,V,\alpha) \cap 2sQ_k \subset 2sQ_k \setminus C_{\mathcal{G}}(x,V,\alpha) \subset 2sQ_k \setminus E \tag{5}$$

Next, we claim that

$$\lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} = 0 \tag{6}$$

We choose $r_k = (\sqrt{2}s + \sqrt{2}/2) \cdot 2^{-k}$ such that apparently $2sQ_k \subset B(x, r_k) \subset 4sQ_k$ for any $x \in Q_k$. Note that by our choice of r_k, when $k \to \infty$, then $r_k \to 0$. By Lemma 2.1 and Corollary A.1,

$$\lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} \leq \lim_{k\to\infty} \frac{\mu(2sQ_k \setminus E)}{K^{-3}\mu(4sQ_k)} \leq K^3 \lim_{k\to\infty} \frac{\mu(B(x,r_k) \setminus E)}{\mu(B(x,r_k))} = 0$$

Therefore (6) holds. Combining (5),

$$\lim_{k\to\infty} \frac{\mu(C_{\mathcal{B}}^{2,1}(Q_k,V,\alpha) \cap 2sQ_k)}{\mu(2sQ_k)} = 0$$

This completes the proof of the necessary condition. □

Note that since $C_{\mathcal{B}}^{2,2}(Q,V,\alpha) \subset C_{\mathcal{B}}^{2}(Q,V,\alpha)$, Lemma A also holds if replace $C_{\mathcal{B}}^{2,1}(Q,V,\alpha)$ with $C_{\mathcal{B}}^{2}(Q,V,\alpha)$ in the limit.

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Appendix A.

In this section we collect the definition, result, and proof that are used above.

Definition A.1 (Diameter)**.** *A non-empty subset A of* $\mathbb{R}^n$ *has the greatest distance apart from points in A, denotes as* $\operatorname{diam}(A)$*. such that*

$$\operatorname{diam}(A) := \sup\{|x-y| : x, y \in A\}$$

Corollary A.1. *If* $A \subset \mathbf{R}^n$ *is* λ *measurable, then the limit*

$$\lim_{r\downarrow 0} \frac{\lambda(A \cap B(x,r))}{\lambda(B(x,r))}$$

exists and equals 1 for λ *almost all* $x \in A$ *and equals 0 for* λ *almost all* $x \in \mathbf{R}^n \setminus A$*.*

The proof can be found in [Mat99, Corollary 2.14].

Proof of Lemma 3.1. (i) When $2R \subset C^{2,1}_{\mathcal{B}}(Q, V, \alpha')$, there exists some $r \in 2R \cap C^1_{\mathcal{B}}(Q, V, \alpha')$ and hence for any $q \in Q$ such that $r \in C_{\mathcal{B}}(q, V, \alpha')$, so $\operatorname{dist}(r-q, V) > \alpha'|r-q|$. Besides, arbitrary $r' \in 2R$ and $q' \in Q$ must satisfy $|r-r'| \leq 2\sqrt{2}\cdot \operatorname{side} R$ and $|q-q'| \leq \sqrt{2}\cdot\operatorname{side} Q$, respectively. Assume that $|r-q| = s'\cdot \operatorname{side} Q$ then $s' \geq s$. Since $|r-r'| \geq |\operatorname{dist}(r'-q', V) - \operatorname{dist}(r-q, V)| - |q-q'|$, no matter $\operatorname{dist}(r'-q', V) \leq \operatorname{dist}(r-q, V)$ or $\operatorname{dist}(r'-q', V) \geq \operatorname{dist}(r-q, V)$,

$$\begin{aligned}
\operatorname{dist}(r'-q', V) &\geq \operatorname{dist}(r-q, V) - |q-q'| - |r-r'| \\
&> \alpha'|r-q| - 3\sqrt{2}\cdot\operatorname{side} Q \\
&\geq (\alpha' s' - 3\sqrt{2})\cdot\operatorname{side} Q \\
&\geq \alpha(s' + 3\sqrt{2})\cdot\operatorname{side} Q \\
&\geq \alpha(|r-q| + |r-r'| + |q-q'|) \\
&\geq \alpha|r'-q'|
\end{aligned}$$

The fourth inequality holds since $s' \geq s \geq \frac{3\sqrt{2}(1+\alpha)}{\alpha'-\alpha}$. Therefore, $2R \subset \{R : R \subset \bigcap_{q' \in Q} C_{\mathcal{B}}(q', V, \alpha)\} = C_{\mathcal{B}}^{2,2}(Q, V, \alpha)$.

(ii) Assume for all $q'' \in 2sQ$, then

$$\begin{aligned} |\operatorname{center} 2R - q''| &\leq \left(\frac{1}{2}\operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2}\operatorname{diam} Q + \frac{1}{2}\operatorname{diam} 2sQ\right) \\ &\leq \left(\sqrt{2} + s + \frac{\sqrt{2}}{2} + \sqrt{2}s\right)\operatorname{side} Q \\ &\leq \frac{2^n}{2}\operatorname{side} R = \frac{1}{2}\operatorname{side} 2^n R \end{aligned}$$

which implies that $2sQ \subset B(\text{center } R, \frac{1}{2}\operatorname{side} 2^n R) \subset 2^n R$.

(iii) Similarly, assume $\forall q''' \in 3sQ$, then

$$\begin{aligned} |\operatorname{center} 2R - q'''| &\leq \frac{1}{2}\operatorname{diam} 2R + \operatorname{dist}(2R, Q) + \frac{1}{2}\operatorname{diam} Q + \frac{1}{2}\operatorname{diam} 3sQ \\ &\leq \left(\sqrt{2} + s + \frac{\sqrt{2}}{2} + \frac{3}{2}\sqrt{2}s\right)\operatorname{side} Q \\ &\leq \frac{2^m}{2}\operatorname{side} R = \frac{1}{2}\operatorname{side} 2^m R \end{aligned}$$

which implies that $3sQ \subset B(\text{center } R, \frac{1}{2}\operatorname{side} 2^m R) \subset 2^m R$.

□

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