Characterization of Rectifiable Measures that are Carried by Lipschitz Graphs

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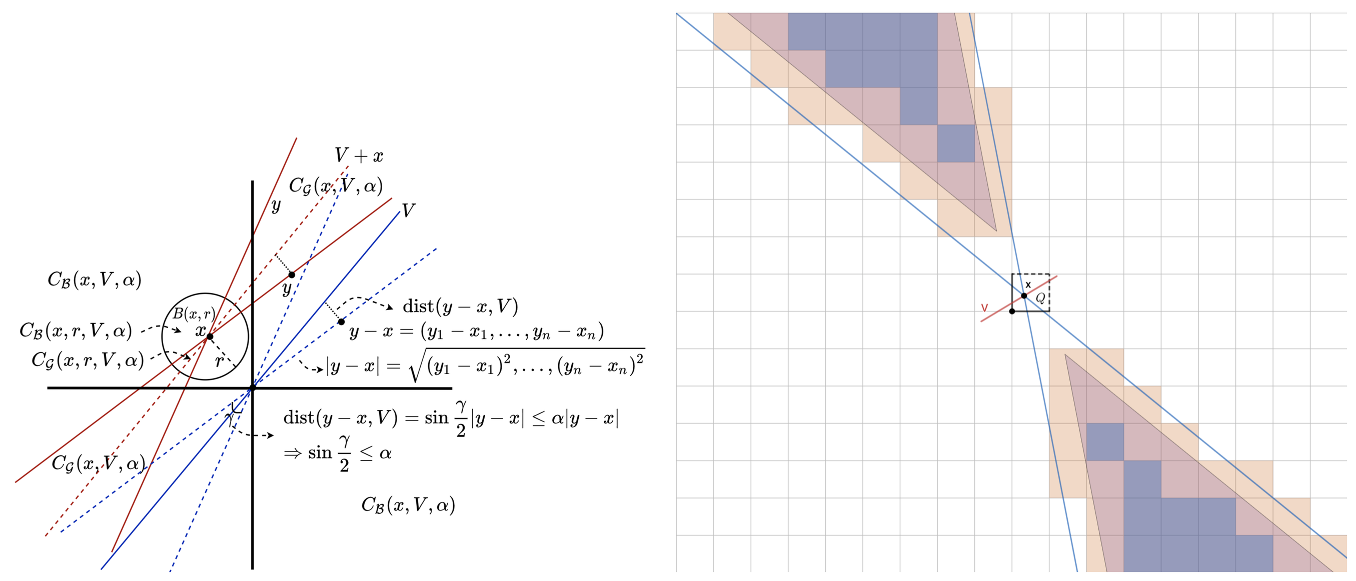
Background

From the analysis perspective,measures are used to define notions of size for sets. For example, traditionally the "size" of a curve is its arc length and the size of a square is its area. To understand how measures assign weight, we explore how they interact with various classes of sets. Here we can explore and find the characterization of measures which are carried by the Lipschitz graphs. Analysts are attracted by the property of Lipschitz graphs that we can find tangents almost everywhere on them.

Definitions

To deep dive into the world of measures, we firstly need to introduce following definitions:

- **Measure** A *measures* $\mu : \mathcal{M} \to \mathbb{R}$ is a function that assigns a "size" to sets in $\mathcal{M}$ according to the following rules:
  - $\mu(E) \geq 0$ for all $E \in \mathcal{M}$
  - $\mu(\emptyset) = 0$.
  - $\mu\left(\bigcup_{i=1}^{\infty} E_i\right) = \sum_{i=1}^{\infty} \mu(E_i)$ for a pairwise disjoint collection of set $\{E_i\}$
- **Doubling Measure** We say that a measure μ is a K-doubling measure for some constant K if for all $r > 0$ and μ-a.e. x,
$$\mu(B(x, 2r)) \leq K\mu(B(x, r))$$
- **Dyadic Cubes** are squares on $\mathbb{R}^2$ formed as cross product of half open intervals with side length as multiples of 1/2.
- **Point Cones** The bad cones on $\mathbb{R}^2$ with respect to a line V through the origin and $\alpha \in (0, 1)$ are defined as
$$C_{\mathcal{B}}(x, V, \alpha) := \{y \in \mathbb{R}^n : \mathrm{dist}(y - x, V) > \alpha|x - y|\}$$
Visualization of point cones and several numerical relationship is illustrated in Figure (b) below.
- **Cube Cones** Bad Cube cones used in our work are defined as $C^I_{\mathcal{B}}(Q, V, \alpha) := \bigcap_{x \in Q} C_{\mathcal{B}}(x, V, \alpha)$, and discretely defined as
$$C^{I,1}_{\mathcal{B}}(Q, V, \alpha) := \{R : R \cap C^I_{\mathcal{B}}(Q, V, \alpha) \neq \emptyset\}$$
$$C^{I,2}_{\mathcal{B}}(Q, V, \alpha) := \{R : R \subset C^I_{\mathcal{B}}(Q, V, \alpha)\}$$
where R is the dyadic cube same as Q. Visualization of three definitions of cube cones are shown in the shaded area in Figure (b) below.



(a) Point Cones (b) Cube Cones

Figure: Cones Visualization

- **Lipschitz Function** The function f is called Lipschitz if
$$|f(x) - f(y)| \leq c|x - y| \quad (x, y \in \mathbb{R}, 0 < c < \infty)$$
- **Lipschitz Graph** Let $f : V \to V^{\perp}$ be a Lipschitz function. Then $\mathrm{Graph}(f) = \{(x, f(x)) : x \in V\}$ is a Lipschitz graph
- **Carried** Let $(\mathbb{R}^2, \mathcal{M})$ be a measurable space, and let $\mathcal{N} \subset \mathcal{M}$ be a family of measurable sets. We say μ is carried by $\mathcal{N}$ if there exist countably many $N_i \in \mathcal{N}$ such that $\mu\left(\mathbb{R}^2 \setminus \bigcup_i N_i\right) = 0$

When we say μ is carried by Lipschitz graphs, it is equivalent that there exists a subset of $\mathbb{R}^n$ is contained μ-a.e. in countably many Lipschitz graph.

Related Work

Geometric Lemma[DL08, Lemma 4.7]

Let $F \subset \mathbb{R}^2$, V be a line through the origin, and $\alpha \in (0, 1)$. If
$$F \cap C_{\mathcal{B}}(x, V, \alpha) = \emptyset \text{ for all } x \in F$$
then $F \subset \Gamma$ where Γ is a 1-Lipschitz graph with respect to V and the Lipschitz constant corresponding to Γ is at most $1 + 1/(1 - \alpha^2)^{1/2}$.

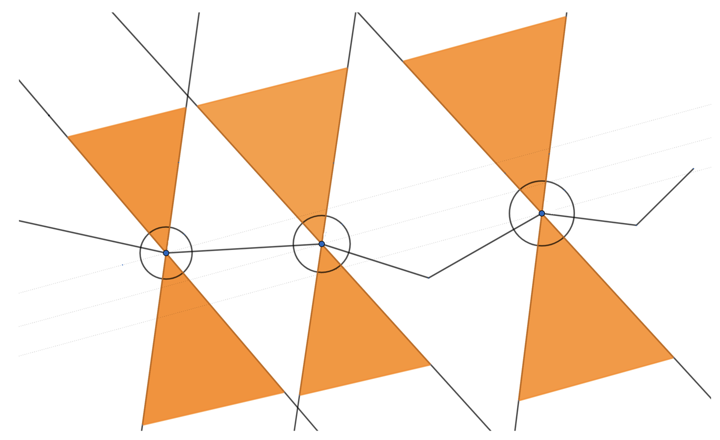


Figure: An example of the geometric lemma

Main Result

Theorem A

Let μ be a K-doubling measure on $\mathbb{R}^2$. Then for μ-a.e. $x \in \mathbb{R}^2$ contained in the dyadic cube Q, there exists a line V, $\alpha \in (0, 1)$, and $s = s(\alpha) \in \mathbb{N}$ such that

- (*Sufficient Condition*) if the limit
$$\lim_{Q \downarrow x} \frac{\mu(C^{I,1}_{\mathcal{B}}(Q, V, \alpha) \cap 2sQ)}{\mu(2sQ)} = 0 \qquad (1)$$
then μ is carried by Lipschitz graphs with respect to V and the Lipschitz constant for each graph is at most $1 + 1/(1 - \alpha'^2)^{1/2}$ for any $\alpha < \alpha' < 1$.
- (*Necessary Condition*) if μ is carried by Lipschitz graphs with respect to V and the Lipschitz constant at most $\alpha/(1 - \alpha^2)^{1/2}$, then (1) holds.

Note: $C^{I,1}_{\mathcal{B}}$ in the limit can be replaced with $C^I_{\mathcal{B}}$ as well.

Proved New Lemmas

To fully prove the sufficient and necessary condition of Theorem A, we first provide lemmas below.

Lemma 1 (Choices of Scaling for Containment)

For $0 < \alpha < \alpha' < 1$, a line V, a dyadic cube Q in $\mathbb{R}^2$, and $s \geq \frac{3\sqrt{2}(1+\alpha)}{\alpha' - \alpha}$. Let $2R$ be a dilation of a cube R such that R is the dyadic cube with the same side length as Q and
$$\mathrm{dist}(Q, 2R) \geq s \cdot \mathrm{side}\, Q$$
then

(i) if $2R \subset C^{I,1}_{\mathcal{B}}(Q, V, \alpha')$, then $2R \subset C^{I,2}_{\mathcal{B}}(Q, V, \alpha)$.

(ii) if for $n = n(s) \in \mathbb{N}$ such that $2^n \geq 2(\sqrt{2} + 1)s + 3\sqrt{2}$, then $2sQ \subset 2^n R$.

(iii) if for $m = m(s) \in \mathbb{N}$ such that $2^m \geq (3\sqrt{2} + 2)s + 3\sqrt{2}$, then $3sQ \subset 2^m R$.

Lemma 2 (Doubling Property for Cubes)

Suppose that a doubling measure μ on $\mathbb{R}^2$ is $K-$doubling, then for any dyadic cube Q containing μ-typical point x,
$$\mu(2^n sQ) \leq K^{3n-3}\mu(2sQ) \qquad (2)$$
where $s, n \in \mathbb{N}, n \geq 2$.

Main Steps of the Proof

Proof of *Sufficient Condition*

Fix a Q_k with a large enough k and construct sets
$$S_{Q_k} := \left\{2R_k : \mathrm{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \mathrm{side}\, Q_k, 2R_k \cap (2sQ_k \cap C^{I,1}_{\mathcal{B}}(Q_k, V, \alpha')) \neq \emptyset\right\}$$
$$S'_{Q_k} := \left\{2R_k : \mathrm{dist}(2R_k, Q_k) \geq \frac{1}{2}s \cdot \mathrm{side}\, Q_k, 2R_k \subset 2sQ_k \cap C^{I,1}_{\mathcal{B}}(Q_k, V, \alpha')\right\}$$
$$S''_{Q_k} := S_{Q_k} \setminus S'_{Q_k}$$
where R_k is a dyadic cube with the same side length as Q_k and $2R_k$ is its dilation. By Lemma 1, Lemma 2, (1), and proof by contradiction we show $\mu(S_{Q_k}) = 0$. With some analysis, we show
$$\mu(C_{\mathcal{B}}(x, r, V, \alpha')) \leq \mu\left(\bigcup_{i=k}^{\infty} \{S_{Q_i} : x \in Q_i \subset Q_k\}\right) = 0$$
By the Geometric Lemma and [Nap20, Corollary 7.1], the sufficient condition holds immediately.

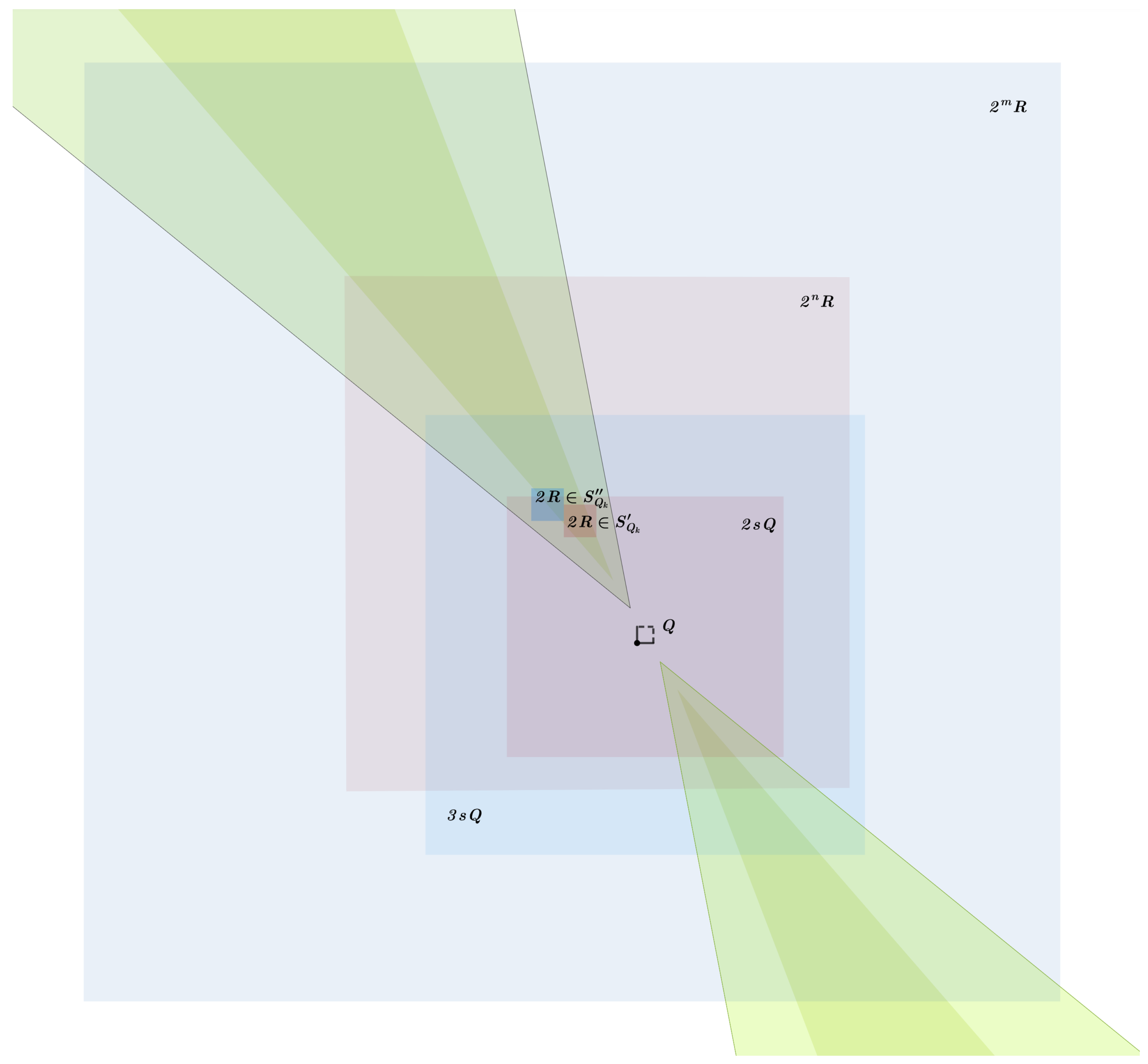


Figure: Visualization of the proof for Sufficient Condition

Proof of *Necessary Condition*

By the assumption, we show
$$\begin{aligned} C^{I,1}_{\mathcal{B}}(Q_k, V, \alpha) \cap 2sQ_k &\subset C_{\mathcal{B}}(x, V, \alpha) \cap 2sQ_k \\ &\subset 2sQ_k \setminus C_{\mathcal{G}}(x, V, \alpha) \subset 2sQ_k \setminus E \end{aligned} \qquad (3)$$
where E consists of μ-a.e. x. By Measure Differentiation [Mat99, Corollary 2.14] and Lemma 2 if we set $r_k = (\sqrt{2}s + \sqrt{2}/2) \cdot 2^{-k}$, we show
$$\lim_{k \to \infty} \frac{\mu(2sQ_k \setminus E)}{\mu(2sQ_k)} \leq \lim_{k \to \infty} \frac{\mu(2sQ_k \setminus E)}{K^{-3}\mu(4sQ_k)} \leq K^3 \lim_{k \to \infty} \frac{\mu(B(x, r_k) \setminus E)}{\mu(B(x, r_k))} = 0 \qquad (4)$$
Combining (3) and (4),
$$\lim_{k \to \infty} \frac{\mu(C^{I,1}_{\mathcal{B}}(Q_k, V, \alpha) \cap 2sQ_k)}{\mu(2sQ_k)} = 0$$
This completes the proof of the necessary condition.

References

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